Time-Stochastic Dominance and iso-elastic discounted utility functions

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Time-Stochastic Dominance and iso-elastic discounted utility functions: a comment on Dietz and Matei (2015)

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Abstract

Dietz and Matei (2015) introduce Time-Stochastic Dominance and apply it to evaluate climate-change mitigation. They compute several preferences classes for which mitigation policies are preferred to business-as-usual. The purpose of the present study is to investigate which standard utility functions (with constant time-discount rate and a constant risk aversion) belong to them. The major contribution is to map preferences classes studied by Dietz and Matei (2015) into the space of time-discount rate and elasticity of marginal utility of consumption, space in which the climate debate has been shaped so far.

Dietz and Matei (2015) introduce the notion of Time-Stochastic Dominance between prospects that involve both a time and a risk dimension. As the related Time Dominance and Stochastic Dominance, Time-Stochastic Dominance between two prospects ensures that the dominant prospect will be preferred to the other for a broad class of time and risk preferences. When the dominance is not exact, Dietz and Matei (2015) introduce almost first-order Time-Stochastic dominance (A1TSD) and construct classes of preferences for which the almost dominant prospect will be preferred. These classes are called $U_1(\epsilon_{1T})$ and $U_1 \times V_1(\gamma_1)$.

Finally, Dietz and Matei (2015) apply A1TSD theory to climate change mitigation. More precisely, they study several stabilisation policies to a ppm target and business-as-usual. None of the stabilisation policies exactly dominates business-as-usual, so that they compute violations of the dominance: some results (a part of their Table 2) are reproduced here. This table should be read as follows: a given stabilisation policy will be prefer to business-as-usual for all preferences that belong to the classes $U_1(\epsilon_{1T})$ and $U_1 \times V_1(\gamma_1)$, where γ_1 and ϵ_{1T} are specific of the policy and given in the table.

Because the preferences classes are complex objects, it is not obvious to decide if a particular utility function belongs to them. As a consequence, it is not straightforward to link the results obtained by Dietz and Matei (2015) to previous works, since the climate debate has been mostly discussed in terms of time-discount rate and elasticity of marginal utility of consumption.

This paper aims at bridging this gap. Its purpose is to find which "standard" utility functions belong to the classes $V_1 \times U_1(\gamma_1)$ and $U_1(\epsilon_{1T})$. A "standard" utility function, as

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Table 2: Violations of exact First-order TSD (reproduced from Dietz and Matei (2015)

CO_2 limit (ppm)	γ_1	ϵ_{1T}
650	0.00009	0.00003
600	0.00045	0.00003
550	0.00092	0.00003
500	0.00188	0.00004
450	0.00388	0.00004

I define it, has a constant time-discount rate and a constant relative-risk aversion. Thus it combines an exponential discounting factor: $v(t) = \mathrm{e}^{-\rho . t}$, ρ being the time-discount rate, and an iso-elastic instantaneous (or CRRA) utility : $u(z) = z^{1-\eta}/(1-\eta)$, η being the elasticity of marginal utility of consumption. Most of the IAM that have a Solow-Ramsey growth model at their core rely on this type of utility function.

We are thus looking for conditions on ρ and η so that the utility function belongs to $V_1 \times U_1(\gamma_1)$ or $U_1(\epsilon_{1T})$. Obviously, these conditions will depend on γ_1 and ϵ_{1T}, but maybe also on some parameters.

The $U_1(\epsilon_{1T})$ class

By definition, a function u in the $U_1(\epsilon_{1T})$ class should satisfied for all $z \in [a, b]$ the following condition:

$$\frac{u'(z)}{\inf[u'(z)]} \leq \frac{1}{\epsilon_{1T}} - 1$$

The interval $[a, b]$ is the actual span of consumption at time T. Because the consumption is stochastic, the interval does not reduce to a point (otherwise the condition will be trivially satisfied). If the initial consumption is between $\underline{C}$ and $\overline{C}$, and the growth rate is between $\underline{g}$ and $\overline{g}$, then at each time the consumption is between $\underline{C}\,\mathrm{e}^{\underline{g}t}$ and $\overline{C}\,\mathrm{e}^{\overline{g}t}$. So the interval $[a, b]$ is $[\underline{C}\,\mathrm{e}^{\underline{g}t}, \overline{C}\,\mathrm{e}^{\overline{g}t}]$.

For an iso-elastic utility, the condition thus becomes

$$\left(\frac{\underline{C}\,\mathrm{e}^{\underline{g}T}}{\overline{C}\,\mathrm{e}^{\overline{g}T}}\right)^{-\eta} \leq \frac{1}{\epsilon_{1T}} - 1$$

or

$$\left(\frac{\overline{C}}{\underline{C}}\right)^{\eta} \mathrm{e}^{\eta \cdot (\overline{g}-\underline{g})T} \leq \frac{1}{\epsilon_{1T}} - 1 \tag{1}$$

We can derive an exact bound for an iso-elastic function of parameter η to be in the class $U_1(\epsilon_{1T})$:

$$\eta \leq \frac{\log\left(\frac{1}{\epsilon_{1T}} - 1\right)}{(\overline{g}-\underline{g})T + \log\left(\frac{\overline{C}}{\underline{C}}\right)} \tag{2}$$

For a given ϵ_{1T}, belonging to the class $U_1(\epsilon_{1T})$ puts more stringent conditions on η when:

- the variability of initial consumption is higher;
- the difference between high growth and low growth is higher;
- the time horizon is longer.

Numerical analysis

To get a better understanding of the order of magnitude involved, I perform a simple numerical analysis.

I assume that there is no initial variability of consumption ($\underline{C} = \overline{C}$), so that all the final variability of consumption comes from the variability of growth. The mitigation policies run from 2015 to 2245 so that $T = 2245 - 2015 = 230$. We set $\epsilon_{1T} = 0.00003$, because most values of Table 2 are close to it.

With a difference between upper and lower growth rates ($\overline{g} - \underline{g}$) of 2%, we find $\eta \lesssim 2.3$. It is quite stringent but still acceptable. This difference has the most impact on the bound on η, because the variability of consumption that comes from it is widened by the long time horizon.

If the difference is expanded to 3%, then η can not be above 1.5. If it is reduced to 1%, then η can become as large as 4.5.

The $V_1 \times U_1(\gamma_1)$ class

By definition, a pair v, u in the class $V_1 \times U_1(\gamma_1)$ should satisfied for all $z \in [a, b]$, and all $t \in [0, T]$:

$$\frac{-v'(t)u'(z)}{\inf[-v'(t)u'(z)]} \leq \frac{1}{\gamma_1} - 1$$

For a standard utility function, we have $-v'(t) = \rho\, \mathrm{e}^{-\rho.t}$ and $u'(z) = z^{-\eta}$. Instead of imposing the condition over the whole rectangle $[0, T] \times [a, b]$, it can be relaxed to a strip that follows consumption over time: $\{(t, z) | 0 \leq t \leq T, a_t \leq z \leq b_t\}$. Only this condition is necessary to obtain the theorem proved in Dietz and Matei (2015). Keeping the same assumptions as in previous analysis, we have that $a_t = \underline{C} e^{\underline{g}t}$ and $b_t = \overline{C} e^{\overline{g}t}$.

To handle this case, we have to make some assumption regarding the growth rates. We suppose that the maximum growth is positive, an assumption that looks reasonable. In this case, the denominator is easily calculated: $\inf[-v'(t)u'(z)] = \rho e^{-\rho.T} \left(\overline{C}\, \mathrm{e}^{\overline{g}.T}\right)^{-\eta}$. Indeed, the infimum is reached at time T for the highest consumption.

Regarding the numerator, it is a little bit more difficult. For the condition to be fulfilled over the consider strip, it is necessary and sufficient that it is satisfied when the numerator is $\rho\, \mathrm{e}^{-\rho.t}\, \underline{C}^{-\eta}\, \mathrm{e}^{-\eta \underline{g}.t}$, for all $t \in [0, T]$. Two cases are now in order. If $\rho + \eta \underline{g} \leq 0$, then it is necessary and sufficient that the condition is satisfied when the numerator is $\rho \underline{C}^{-\eta}$. When $\rho + \eta \underline{g} < 0$, this condition is still necessary but no longer sufficient: it is necessary and sufficient for the condition to be satisfied when the numerator is $\rho\, \mathrm{e}^{-\rho.T}\, \underline{C}^{-\eta}\, \mathrm{e}^{-\eta \underline{g}.T}$. To simply the numerical analysis, we will only retain in the sequel the condition when the numerator is equal to $\rho \underline{C}^{-\eta}$. This will thus provide a necessary condition for a standard utility function to belong to the class $V_1 \times U_1(\gamma_1)$. We have to keep in mind that, when $\underline{g} < -\rho/\eta$ (an unlikely but possible outcome), this condition will not be sufficient so that the set of standard utilities belonging to the class will actually be smaller.

Therefore, a necessary condition to belong to the class $V_1 \times U_1(\gamma_1)$ is:

$$\frac{\underline{C}^{-\eta}}{e^{-\rho.T} \left(\overline{C}\, \mathrm{e}^{\overline{g}.T}\right)^{-\eta}} \leq \frac{1}{\gamma_1} - 1$$

It can be rewritten:

$$\left(\frac{\overline{C}}{\underline{C}}\right)^{\eta} \mathrm{e}^{(\rho + \overline{g}.\eta)T} \leq \frac{1}{\gamma_1} - 1$$

For a given γ_1, this condition becomes more stringent when:

- the variability of initial consumption is higher
- the maximum growth is larger
- the time horizon is longer

We finally obtain:

$$\rho.T + \eta.\left(\overline{g}T + \log\left(\frac{\overline{C}}{\underline{C}}\right)\right) \leq \log\left(\frac{1}{\gamma_1} - 1\right)$$

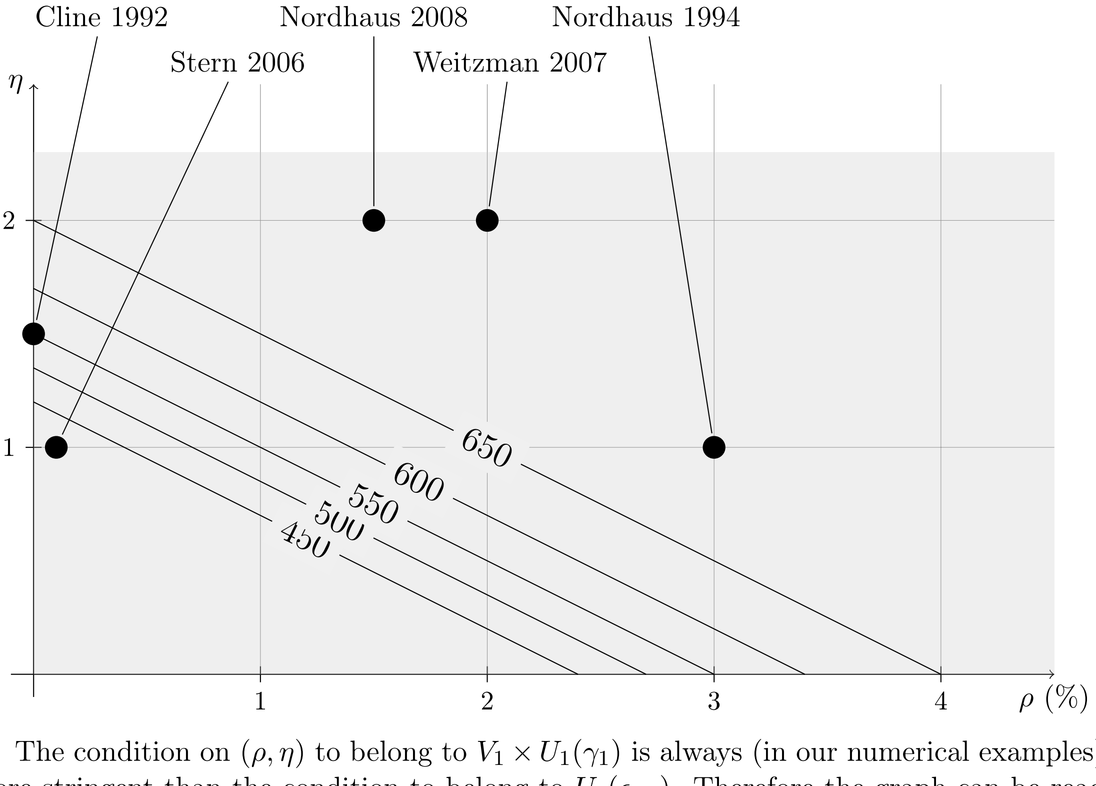
In the (ρ, η) plane, the class $V_1 \times U_1(\gamma_1)$ is under a straight line of slope approximately $-1/\overline{g}$, when T is large.

Numerical analysis

As before, I assume no variability of initial consumption ($\frac{\overline{C}}{\underline{C}} = 1$) and $\overline{g} = 2\%$ (which is maybe a little bit small), and thus, to be consistent with previous analysis $\underline{g} = 0$.

The following graph displays the classes studied so far in the (ρ, η) plane. The class $U_1(\epsilon_{1T})$ is the grey-shaded area. The class $V_1 \times U_1(\gamma_1)$ is depicted in the following graph with its boundary line, for each γ_1 of Table 2. The class is thus the triangle below the downward sloping line. Each line is labelled by the ppm limit of Table 2, to make the correspondence easier with the policy.

Figure 1: In the plane of time-discount rate ρ and elasticity of marginal utility of consumption η: class $U_1(\epsilon_{1T}$ (in grey) and classes $V_1 \times U_1(\gamma_1)$ of Table 2; emblematic stands of the climate debate.



The condition on (ρ, η) to belong to $V_1 \times U_1(\gamma_1)$ is always (in our numerical examples) more stringent than the condition to belong to $U_1(\epsilon_{1T})$. Therefore the graph can be read as follows: if a pair (ρ, η) is chosen under a line marked by xxx ppm, then, with the chosen utility function, the stabilisation policy at xxx ppm will be preferred to business-as-usual.

On the graph, several emblematic stands of the climate debate are also represented. With the numbers computed here, several stands that support a high-discount rate do not belong to any preferences class computed by Dietz and Matei (2015).

A word of caution: when parameters of a utility function are above the xxx ppm line, it does not mean necessarily that the business-as-usual is preferred to the stabilisation policy. It is just that the A1TSD theory cannot be applied to conclude that the stabilisation policy is preferred to the business-as-usual.

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In this short note, I have made a tentative analysis to relate standard utilities functions to the preferences classes defined by Dietz and Matei (2015). A more precise estimation should use the real numbers (regarding $\frac{\overline{C}}{\underline{C}}$, $\overline{g}$ and $\overline{g} - \underline{g}$) delivered by the DICE implementation used therein.

The analysis delivers interesting orders of magnitude on parameters of standard utility functions that belong to the classes that related to Almost First-order Time-Stochastic Dominance.

References

Cline, William R. (1992) — *The economics of Global Warming*, Washington: Institute for International Economics.

Dietz, Simon and Nicoleta Anca Matei (2015) — Spaces for agreement: a theory of Time-Stochastic Dominance and an application to climate change, to be published in *Journal of the Association of Environmental and Resource Economists.*

Nordhaus, William D. (1994) — *Managing the Global Commons: The Economics of Climate Change*, Cambridge (Mass.): MIT Press.

Nordhaus, William D. (2008) — *A Question of Balance*, London: Yale University Press.

Stern, Nicholas H. (2006) — *Stern Review: The Economics of Climate Change*, Volume 30, London: HM treasury.

Weitzman, Martin L. (2007) — A review of "the Stern Review on the economics of climate change", *Journal of Economic Literature* ***45***(3), pp. 703–724.